Probability and Six-Dice Zilch

Binary System

In a Binary System, only two Outcomes are possible -- as in "Heads" or "Tails" in a coin toss. For these Zilch Probability calculations, the only two Outcomes are:

Probability of Success	**Ps**	(Chance of a "One", a "Five" or 3-of-a-Kind or of matching a 3-of-a-Kind on the Board)
Probability of Failure	**Pf**	(Chance of not achieving above lofty goals]

As these are the only possible outcomes, logic follows:

Ps + Pf = 1 (1 = 100% = one hundred per one hundred)
Ps = 1 - Pf

The lower formula comes into play when the Probability of Failure is simpler to calculate, which is usually, though not always, the case -- as we shall see.

Calculating Ps [1]

Read as Probability of Success with **one** Die and **no** Triplets of 2's, 3's, 4's, or 6's on the Board. Success in this case means rolling a "One" **or** a "Five", and since **either** Outcome amounts to a Success, we **add** the Probabilities of each Outcome, hence:

P [1](1) **= 1/6** (Read as Probability of rolling, with one Die, a "One")
P [1](5) **= 1/6** (Read as Probability of rolling, with one Die, a "Five")

Ps [1] **= P [1](1) + P [1](5)**
= 1/6 + 1/6 = 1/3

Calculating Pf [1]

Failure in this case means rolling a "Two", a "Three", a "Four", **or** a "Six", and since **any** of these four Outcomes brings upon Failure, again we **add** the individual Probabilities of the four Outcomes:

Pf [1] **= P [1](2) + P [1](3) + P [1](4) + P [1](6)**
= 1/6 + 1/6 + 1/6 + 1/6 = 2/3

We formally did these calculations, when we could have observed that 2 of 6 sides Succeed, whereas 4 of 6 sides Fail, and been done with it, but we need to recognize conditions wherein we **add** Probabilities ...and when we **multiply** them -- more on this in next section. Also, we observe that indeed **Ps + Pf = 1**.

Let's look at a simple problem with Overlap: Calculate the Probability of tossing at least one "Head" with two coins. A first stab may be to double the Probability of tossing a "Head" with one coin: 2 x 50% = 100%, which is illogical. The wrinkle here is that "Heads and Heads" got double-counted, each coin claiming the "victory" as its own. So we subtract one Chance of double "Heads": 100% - 25% = 75%, which *is* logical.

We now contend with two Dice and more complications in our calculations for Probability of Success. When calculating Probabilities for Outcome Alpha **or** Outcome Beta to Succeed/Fail, we **add** the individual Probabilities -- and subtract the Overlap, which was "Zero" in above section's Probability. Again, since **either** a "One" **or** a "Five" Succeeds on **either** Die, we again **add** the Probabilites, but here we need to contend with Overlap. Let's layout the first attempt...tackle the Overlap…then redo whole thing simpler.

Ps [2] = P [2](1) + P [2](5)
= P [1](1) + P [1](1) + P [1](5) + P [1](5)
= 1/6 + 1/6 + 1/6 + 1/6
= 2/3

The seasoned Zilch player will point out that we do not Succeed 67% of the time with two Dice…so what's up? The problem is that we inadvertently counted eight Successes when only *four* were actual Successes, those being **1-1, 1-5, 5-1, 5-5**. Confused enough? Let's layout the sacred Outcome Table.

[The Page 3 Outcome Table is downloadable from the Probability Page.]

We double-counted the Overlap, so now we calculate the Probability of "Double Succeeders" with two Dice and subtract. We need the combined Probability of Success on one die **and** the other, so we **multiply:**

Ps [1] AND Ps [1]	**= Ps [1] x Ps [1]**	
Overlap	**= 1/3 x 1/3 = 1/9**	(Outcomes number 36, and 4 Overlaps/36 = 1/9)

Ps [2]	**= 2/3 - Overlap**	
	= 2/3 - 1/9	
	= 5/9	(Outcomes of 36 squares indeed yields 20 Successes)

Calculating Pf [2]

Calculating the Chances of Success and the Overlap gets convoluted quickly as Dice-count increases. But the calculation for Failure *can* be more straightforward. Here, with **Pf [2]**, one Die **and** the other must Fail:

Pf [2] =	**= Pf [1] x Pf [1]**
	= 2/3 x 2/3 = 4/9
Ps [2]	**= 1 - Pf [2]**
	= 1 - 4/9 = 5/9

Elegant and simple, and we avoid the Overlap conundrum. But sometimes a combination is called for…

Calculating Ps [3]

Suppose we front-assault and directly calculate **Ps [3] {with no 3-of-a-Kind on Board}**. Well, let's see:

Ps [3]	**= Ps [1] + Ps [1] + Ps [1] + P [3]{--> 3} - Overlap**

The **P [3] {--> 3}** *can* be calculated (Probability with three Dice for obtaining a 3-of-a-Kind)…and in fact we *will* work that out, but let's *not* attempt the Overlap -- brutal. Instead, let us zig-zag a bit as we pursue the Probability of Failure with three Dice. To Fail, **all** three Dice need to Fail, so that's an **and**, but to this **product** must also be **multiplied** the Probability of Failure to achieve a 3-of-a-Kind:

Pf [3]	**= Pf [1] x Pf [1] x Pf [1] x Pf [3]{--> 3}**
	= 2/3 x 2/3 x 2/3 x Pf [3]{--> 3}
	= 8/27 x Pf [3]{--> 3}

Terrific…how the heck do we calculate the Probability of Failure to get a 3-of-a-Kind? Grueling. But! then we realize it's easier to calculate the Probability to *get* a 3-of-a-Kind, and then Subtract *that* Chance from One, and so derive our Probability of Failure to get a 3-of-a-Kind. The Probabilities of Success Space of a 3-of-a-Kind are Exclusive (Overlap equals "Zero"), and as **any** of the four Triplets will result in a Success, the Probabilities for a 3-of-a-Kind can be simply **added:**

P [3]{-->3}	**= P [3](2's) + P [3](3's) + P [3](4's) + P [3](6's)**	
	= (1/6)^3 + (1/6)^3 + (1/6)^3 + (1/6)^3	[(1/6)^3 = 1/6 x 1/6 x 1/6]
	= 4 x (1/6)^3 = 4 /(9 x 4 x 6) = 1/(9 x 6) = 1/54	

Pf [3]{--> 3} **= 1 - P [3]{--> 3}**
= 1 - 1/54 = 53/54

Pf [3] **= 8/27 x Pf [3]{--> 3}**
= 8/27 x 53/54 = 0.291 [What! Everything was so exact till now!]

Ps [3] **= 1 - Pf [3]**
= 1 - 3/10
= 7/10

Calculating Ps [1]{3}

Read as Probability of Success with **one** Die and a **Triplet** of either 2's, 3's, 4's or 6's on the Board. This may be easier for us to conceive if we were to settle on a supposed Trio, say "Sixes". We'll address this straightforward. With one Die, our Success Space consists of **either** a "One," a "Five," **or** a "Six":

Ps [1]{3} **= P [1](1) + P [1](5) + P [1](6)**
= 1/6 + 1/6 + 1/6 = 1/2

Calculating Ps [2]{3}

We now have two Dice and let us stay with our supposed Trio of "Sixes". By now, we are properly leary of the Overlap hastle, so let us shun the angst once again by calculating the Probability of Failure. Since **all** Outcomes need to Fail in order to Fail, **both** Dice must exhibit **either** a "Two", a "Three" **or** a "Four":

Pf [2]{3} **= Pf [1] x Pf [1]**
= [P [1](2) + P [1](3) + P [1](4)]^2
= [1/6 + 1/6 + 1/6]^2 = 1/4

We could've noticed that half of Outcomes Fail, and then squared the half for **Pf [2]{3}**.

Ps [2]{3} **= 1 - Pf [2]{3}**
= 1 - 1/4 = 3/4

Note that if we calculated by **Ps [2]{3} = Ps [1]{3} + Ps [1]{3}** without subtracting Overlap, we get 100%.

Calculating Ps [3]{3}

So, let's calculate our last pragmatic Probability, for *who* wouldn't throw four Dice? Now, here we have three Dice, possible matches with Trio of **either** 2's, 3's, 4's, or 6's on the Board, ***plus*** the Probability of creating another 3-of-a-Kind. Again, supposing three "Sixes", we note that here **Pf [1](1, 5 or 6) = 1/2** .

Ps [3]{3} **= 1 - Pf [3]{3}**

Pf [3]{3} **= Pf [1] x Pf [1] x Pf [1] x Pf [3]{--> 3}**
= 1/2 x 1/2 x 1/2 x Pf [3]{--> 3}

Pf [3]{--> 3} **= 1 - P [3]{--> 3}**

Calculating the Probability of creating a 3-of-a-Kind with three Dice, as was done before, except that we leave out the P [3](6's) since the Probability of rolling a "Six" is already calculated:

P [3]{--> 3} = P [3](2's) + P [3](3's) + P [3](4's)
= (1/6)^3 + (1/6)^3 + (1/6)^3
= 3 x (1/6^3) = 3 /(4 x 9 x 6) = 1/72

Pf [3]{--> 3} = 1 - P [3]{--> 3}
= 1 - 1/72 = 71/72

Pf [3]{3} = 1/2 x 1/2 x 1/2 x Pf [3]{--> 3}
= 1/8 x 71/72
= 0.123 = 1/8

Ps [3]{3} = 1 - Pf[3]{3}
= 1 - 1/8 = 7/8

Calculating Pf [6]

Doesn't it seem like a six-Dice Zilch occurs more often than it should? Let's see if we can calculate the Probability of Failure with six Dice: **all** Fail having a "One" or a "Five" **and** Fail to garner a 3-of-a-Kind:

Pf [6] = Pf [1]^6 x Pf [6]{--> 3}
= (2/3)^6 x Pf [6]{--> 3}

Pf [6]{--> 3} = 1 - P [6]{--> 3}

Here we enter the world of Permutations, combinations of three Dice out of the six thrown, and which in fact *can* be calculated with established formulas. But let us keep it simple and divide the six Dice into two groups of three, so that we can then use the **P [3]{--> 3}** that we already worked out, and double *that* Probability. We will slightly underestimate the true Probability of creating Trios, but let's do it anyway:

P [6]{--> 3} = 2 x P [3]{--> 3}
= 2 x 4 x (1/6)^3
= 2 x 1/54 = 1/27

Pf [6]{--> 3} = 1 - P [6]{--> 3}
= 1 - 1/27 = 26/27

Pf [6] = (2/3)^6 x Pf [6]{--> 3}
= (2/3)^6 x 26/27
= 8/100

Even when we triple our estimate of **P [6]{--> 3}** to 3/27 or 1/9, the above calculations render the same round-offed 8/100, so we're confident here. Seems about right, right? Doubters of this result/method are *encouraged* to directly calculate the Probability of Success with six Dice. But watch out for the Overlap! (An annoying Zilch player -- Pat -- would point out non "One/Five" Three-Pair's: **Ps [6] = 92% + 1% = 93%**)

Calculating Ps [6](of at least three "Ones" for 1000 points)

After considerable head-scratching, we simplified the Problem to finding the Probability of two "Ones" **and** the Probability of at least a third "One" among the remaining four Dice. Here's where it gets "dicey". Calculating Probability with 4 Dice to obtain a "One" is simple: **P [4](1) = 1 - Pf [4](1) = 1 - (5/6)^4 = 52%** . But how about first Probability of *first* two "Ones"? And again keeping it simple (and probably getting it wrong), let's brashly march the connected pair across the six Dice and count the occasions:

```
1 1 0 0 0 0
0 1 1 0 0 0
0 0 1 1 0 0
0 0 0 1 1 0
0 0 0 0 1 1
```

(Not sure why the pairs have to say connected, but at least for this approach, seems logical.)
We count five occurrences, and since **any** occurrence Succeeds, we **add** them, or multiply by 5:

Ps [6](at least three "Ones") = 5 x P [2](a "One" and a "One") x P [4](at least a "One")
= 5 x (1/6 x 1/6) x (1 - Pf [4](1))
= (5/36) x (1 - (5/6)^4)
= 0.07

Visit to "omnicalculator.com" yielded a Probability of 6% but did not show calculations. We like our 7%. The point to be made here is that the calculations get hairy fast with six Dice and a more experienced mathematician is often needed to enter the scene -- more going on here. And the prickly feeling is that we got lucky here figuring the Probability of first two "Ones" appearing. But not bad for Amatuers!

Table of Probability of Success in Six-Dice Zilch

Dice	No Trio	With Trio
1	**33%**	**50%**
2	**56%**	**75%**
3	**71%**	**88%**
6	**93%**	
6 (3 "Ones")	**7%**	